

Probability in relativistic Bohmian mechanics of particles and strings

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Abstract

Even though the Bohmian trajectories given by integral curves of the conserved Klein-Gordon current may involve motions backwards in time, the natural relativistic probability density of particle positions is well-defined. The Bohmian theory predicts subtle deviations from the statistical predictions of more conventional formulations of quantum theory, but it seems that no present experiment rules this theory out. The generalization to the case of many particles or strings is straightforward, provided that a preferred foliation of spacetime is given.

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1 Introduction

The Bohmian interpretation of nonrelativistic quantum mechanics (QM) [1, 2, 3] is the best known and most successful reformulation of QM in terms of hidden variables. However, the generalization to the relativistic case is still an unsettled issue. One of the approaches is the most direct generalization based on 3 natural steps: (i) the nonrelativistic Schrödinger equation is replaced by the corresponding relativistic wave equation, (ii) the conserved probability current (associated with the Schrödinger equation) is replaced by the appropriate relativistic conserved current associated with the relativistic wave equation, and (iii) the relativistic Bohmian particle trajectories are postulated to be the integral curves of this relativistic conserved current. However, the simplest generalization consisting of these 3 natural steps is not without difficulties. For example, in [3] it has been considered problematic because for bosonic particles such Bohmian velocities may be superluminal. Nevertheless, by using the theory of quantum measurements, it has been stressed that measured velocities cannot be superluminal [4, 5], which avoids possible clashes with observations. Similarly, such Bohmian particles can move backwards in time leading to multiple particle positions at a single time, but again, by using the theory of quantum measurements it has been argued that such multiple positions cannot be observed [5].

Another objection against such a version of the Bohmian interpretation is based on the fact that the time component of the conserved current may be negative, which means that it cannot be interpreted as the probability density of particle positions. In [5] it was argued that it

does not necessarily make the Bohmian interpretation inconsistent, because at the fundamental level Bohmian mechanics is a fully deterministic theory, so it does not need to have an *a priori* defined probability density. Instead, the results in nonrelativistic QM [6] suggest that a simple relation between wave function and probability density may be an emergent phenomenon, not a fundamental one. Nevertheless, the absence of an *a priori* probability density of particle positions makes the theory less predictive, so even if it is not a fundamental problem, it makes the theory less useful in practice.

In this paper we show that the natural conserved probability density *can* be introduced. Essentially, it is the absolute value of the time-component of the conserved current, but in the single-particle case it is fully relativistic-covariant. In the n -particle case it requires a preferred foliation of spacetime, but the foliation can be specified by a unit vector normal to the preferred hypersurfaces which allows to write all the equations in a relativistic-covariant form. It can also be further generalized to the case of strings. The nontrivial aspect of this probability density stems from motions backwards in time and multiple particle positions, implying that a hypersurface on which the total probability is equal to one may not be spacelike everywhere or may be given by only a part of a spacelike Cauchy hypersurface. Nevertheless, such hypersurfaces are defined by the congruence of the integral curves of the conserved current, so in principle all statistical predictions are uniquely defined (up to the choice of preferred foliation). As demonstrated in [5], the measurable predictions may differ from those of more conventional formulations of quantum theory, but we argue that no present experiment rules this Bohmian theory out.

The next section deals with the single-particle case, while Secs. 3 and 4 contain the generalizations to the cases of many particles and strings, respectively. The conclusions are drawn in Sec. 5. We use the units $\hbar = c = 1$ and the metric signature $(+, -, -, -)$.

2 Probability in the single-particle case

2.1 General theory

For the sake of brevity, this subsection is *not* intended to be self-contained. Instead, we extensively use some mathematical results and geometrical insights explained in more detail in [5] and [7]. The main line of reasoning can be followed without explicit reference to these papers, but for the sake of more complete understanding we recommend to consult these papers as well.

Let $\hat{\phi}(x)$ be a scalar hermitian field operator satisfying the free Klein-Gordon equation. If $|0\rangle$ is the vacuum and $|1\rangle$ is an arbitrary 1-particle state, the corresponding c-number valued wave function is $\psi(x) = \langle 0|\hat{\phi}(x)|1\rangle$ (see also the Appendix). Such $\psi(x)$ is a superposition of positive-frequency solutions of the Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\psi(x) = 0. \quad (1)$$

The naturally associated conserved current is the Klein-Gordon current

$$j_\mu = i\psi^* \overleftrightarrow{\partial}_\mu \psi. \quad (2)$$

It is normalized so that

$$\int_\Sigma dS^\mu j_\mu = 1, \quad (3)$$

where Σ is an arbitrary 3-dimensional spacelike Cauchy hypersurface, $dS^\mu = d^3x |g^{(3)}|^{1/2} n^\mu$ is the covariant measure of the 3-volume on Σ , n^μ is the unit future-oriented vector normal to Σ ,

and $g^{(3)}$ is the determinant of the induced metric on Σ . The conservation equation

$$\partial_\mu j^\mu = 0 \quad (4)$$

provides that (3) does not depend on Σ .

In the Bohmian interpretation, the particles have trajectories given by integral curves of the conserved vector field $j^\mu(x)$. These curves can be parametrized by an auxiliary affine scalar parameter s , so that the explicit Bohmian equation of motion reads

$$\frac{dx^\mu(s)}{ds} = j^\mu, \quad (5)$$

which determines the trajectory $x^\mu(s)$.

If $j^0(x)$ is non-negative for all x , then (4) implies that $j^0(x)$ is the natural conserved probability density of particle positions at a hypersurface of constant x^0 . Such a probability density is compatible with the Bohmian equation of motion (5). More generally, if

$$\tilde{j} \equiv |g^{(3)}|^{1/2} n^\mu j_\mu \quad (6)$$

is non-negative for all x for every timelike future-oriented n^μ , then $\tilde{j}(x)$ is the natural conserved probability density of particle positions at an arbitrary spacelike hypersurface specified by its unit normal $n^\mu(x)$. The results of [7] show that this relativistic-covariant definition of particle density is also compatible with (5).

The non-trivial issue is to generalize this to the case in which j^0 , or more generally \tilde{j} , may be negative at some x . Nevertheless, the generalization is rather simple; to make the probability density non-negative, one simply has to take the absolute value of j^0 [8] or \tilde{j} . Indeed, the absolute value $|j^0|$ also satisfies a *local* conservation equation of the form $\partial_0 |j^0| \pm \partial_i j^i = 0$, where the upper (lower) sign is valid for x at which j^0 is positive (negative). Thus, in general, the local conserved probability density is simply

$$\tilde{p}(x) = |\tilde{j}(x)|. \quad (7)$$

The non-trivial aspect of (7) is the correct *global* interpretation of it. In general, from (3) we see that

$$\int_\Sigma d^3x |\tilde{p}| = \int_\Sigma dS^\mu |j_\mu| \geq 1, \quad (8)$$

while, according to the standard theory of probability, the sum of probabilities for all possibilities of particle positions that constitute the sample space should be strictly equal to 1, not ≥ 1 . Nevertheless, the interpretation of this apparent inconsistency is rather simple: owing to the deterministic motions of particles described by (5), not all possibilities of particle positions on Σ count as different elements of the sample space. More precisely, if a trajectory crosses Σ at a point x_A , then the same trajectory may cross the same Σ at another point x_B , in which case x_A and x_B represent the same element of the sample space. Indeed, as discussed in more detail in [4, 5], and mathematically more rigorously in [8], this is a direct consequence of the fact that j^μ may be spacelike and j^0 may be negative at some regions of spacetime. Therefore, instead of using the timelike Cauchy hypersurface Σ in (8), one must use a different 3-dimensional hypersurface Σ' , chosen such that *no trajectory crosses Σ' more than once*. On such a hypersurface one has

$$\int_{\Sigma'} d^3x |\tilde{p}| \leq 1. \quad (9)$$

If Σ' is such that (9) is equal to 1, then we say that Σ' is *complete*. The case < 1 in (9) may occur because Σ' may be chosen such that some trajectories never cross Σ' . Typically, the case < 1 corresponds to the case in which Σ' is a subset of the Cauchy hypersurface Σ [5].

In practice, it is generally nontrivial to find a complete Σ' . This is because, in principle, one needs to know the whole congruence of integral curves of the vector field $j^\mu(x)$, i.e., all the trajectories in the whole spacetime. Nevertheless, for given $\psi(x)$ this is well defined in principle (and straightforward to find numerically). There are two typical shapes that such complete hypersurfaces Σ' may take. First, one may require that Σ' should be connected. In this case some regions of Σ' may not be spacelike [7]. Second, one may require that Σ' should be spacelike everywhere. In this case Σ' may not be connected [5]. (Such a disconnected Σ' consists of 2 or more mutually disconnected pieces, each being a connected subset of Σ .) Of course, a mixture of these two typical shapes, i.e., a hypersurface which is neither connected nor spacelike everywhere, is also possible.

It is also of interest to know about the cases in which the trajectories do not necessarily need to be calculated explicitly. From the results of [5] one can infer the following: If Σ' is a connected subspace of the spacelike Cauchy hypersurface Σ such that \tilde{j} has the same sign everywhere on Σ' , then no trajectory crosses Σ' more than once. Consequently, such connected spacelike Σ' with constant sign of \tilde{j} can be used in (9).

So far we have been tacitly assuming that ψ satisfies the free Klein-Gordon equation everywhere and that $j^\mu(x)$ is a regular vector field everywhere. However, when the detection process or the initial creation of particles is taken into account, this does not longer need to be the case. In particular, owing to the motions backwards in time, not all trajectories need to cross (not even ones) every spacelike Cauchy hypersurface Σ . For example, it may happen that the particle created at t_0 never reaches the detector starting with operation at $t_1 > t_0$. Another possibility is that some trajectories become completely unphysical, in the sense that no initial condition corresponding to a particle existing at t_0 is compatible with these trajectories [5]. This leads to an interesting prediction that, instead of multiple particle positions, one will actually observe that a particle will never be found at some positions at which the wave function does not vanish [5]. (In such cases, the probability density on different regions of spacetime is either given by (7) or equal to zero, which, as demonstrated in [5], is also determined by the whole congruence of integral curves induced by $j^\mu(x)$.)

2.2 On measurable consequences

As we have seen, negative values of j_0 are related to motions backwards in time, which may lead to multiple particle positions at the same time. This, of course, can be interpreted as particle creation. However, since this occurs even for free particles, such a prediction of particle creation does not coincide with predictions on particle creation in more conventional formulations of quantum theory. (Note, however, that *the* standard prediction of probabilities of particle positions does not really exist, because the standard approach requires a relativistic position operator, which does not exist. Therefore, the issue of probabilities of relativistic particle positions is an unsettled issue even within the conventional formulations of quantum theory [9].) Owing to the difference between the predictions of the Bohmian and the conventional formulation, one could jump to the conclusion that this makes such a Bohmian interpretation untenable. A more optimistic view is that this difference could be used to test the Bohmian formulation experimentally. However, in this subsection we argue that it is actually rather difficult to see the differences in practice and that probably no currently existing experiment can be used to rule out such a version of the Bohmian interpretation.

First, most experiments on relativistic particles are based on scattering experiments. Such experiments are better viewed as measurements of particle momenta (rather than positions), in which case the predictions of the Bohmian interpretation coincide with those of the standard interpretation [4].

Next, experiments that measure quantum probabilities of particle positions (e.g. measurements of interference patterns) do exist, but they are usually based on stationary wave functions, namely functions of the form

$$\psi(x) = \frac{e^{-i\omega t}}{\sqrt{2\omega}} \varphi(\mathbf{x}). \quad (10)$$

For such wave functions

$$j_0(\mathbf{x}, t) = |\varphi(\mathbf{x})|^2, \quad (11)$$

so the probability density of particle positions coincides with that of a more conventional formulation. More generally, whenever j_0 is non-negative everywhere one may view j_0 as a conventional probability density, because in conventional views j_0 can be identified with charge density, so for a neutral particle it is reasonable to expect that it coincides with the probability density.

Thus, to obtain a prediction of the Bohmian interpretation that differs from conventional ones, we must deal with a case in which j_0 may be negative. The necessary (though not sufficient) condition is that the state should be a superposition of two or more different frequencies. Therefore, let us study the case of two different equally probable frequencies

$$|1\rangle = \frac{|k_1\rangle + |k_2\rangle}{\sqrt{2}}, \quad (12)$$

where $|k_1\rangle$ and $|k_2\rangle$ are the 4-momentum eigenstates with 4-momenta k_1 and k_2 , respectively. The corresponding wave function (normalized in a finite 3-volume V) is

$$\psi(x) = \frac{1}{\sqrt{2}} \left[\frac{e^{-ik_1 \cdot x}}{\sqrt{V2\omega_1}} + \frac{e^{-ik_2 \cdot x}}{\sqrt{V2\omega_2}} \right], \quad (13)$$

where $\omega_{1,2} = \sqrt{\mathbf{k}_{1,2}^2 + m^2}$. Thus (2) gives

$$j^\mu(x) = \frac{1}{2V} \left[\frac{k_1^\mu}{\omega_1} + \frac{k_2^\mu}{\omega_2} + \frac{k_1^\mu + k_2^\mu}{\sqrt{\omega_1\omega_2}} \cos[(k_1 - k_2) \cdot x] \right]. \quad (14)$$

We know that the non-relativistic limit leads to non-negative j^0 , so the most interesting case is expected to be the ultrarelativistic limit $m \rightarrow 0$. Therefore, we study the case $m = 0$. For simplicity, we study the case of 1+1 dimensional motion, i.e., we assume that $k_{1,2}^\mu$ is nonvanishing only for $\mu = 0, 1$. Thus, the momenta k_1 and k_2 are either collinear (the space components of momenta have the same directions) or anti-collinear (the space components of momenta have the opposite directions).

First consider the case of collinear momenta. Thus we take $k_1^1 = \omega_1$, $k_2^1 = \omega_2$ (the upper label means $\mu = 1$), so the non-vanishing components of (14) are

$$j^0 = \frac{1}{V} \left[1 + \frac{\omega_1 + \omega_2}{2\sqrt{\omega_1\omega_2}} \cos[(\omega_1 - \omega_2)(t - x^1)] \right], \quad (15)$$

$$j^1 = \frac{1}{V} \left[1 + \frac{\omega_1 + \omega_2}{2\sqrt{\omega_1\omega_2}} \cos[(\omega_1 - \omega_2)(t - x^1)] \right]. \quad (16)$$

We see that (15) is negative for some x , provided that $\omega_1 \neq \omega_2$. Nevertheless, we see that $j^1 = j^0$, which means that the trajectories satisfy $dx^1/dx^0 = j^1/j^0 = 1$, i.e. all trajectories have a constant velocity equal to the velocity of light. In other words, even though j^0 is negative for some x , there are no motions backwards in time and thus there are no measurable deviations from more conventional formulations of quantum theory.

Now consider the case of anti-collinear momenta. Thus we take $k_1^1 = \omega_1$, $k_2^1 = -\omega_2$, which leads to

$$j^0 = \frac{1}{V} \left[1 + \frac{1+\eta}{2\sqrt{\eta}} \cos[(1-\eta)\omega_1 t - (1+\eta)\omega_1 x^1] \right], \quad (17)$$

$$j^1 = \frac{1}{V} \left[\frac{1-\eta}{2\sqrt{\eta}} \cos[(1-\eta)\omega_1 t - (1+\eta)\omega_1 x^1] \right], \quad (18)$$

where $\eta \equiv \omega_2/\omega_1$. Now j^0 is negative for some x and $j^0 \neq j^1$, so motions backwards in time are possible. (We have confirmed that by numerically finding the trajectories determined by (17)-(18).) However, in practice, it seems to be very difficult to measure particle positions for such a state. Namely, this state corresponds to a superposition of two coherent beams moving in the opposite directions, so they cannot both hit the detection screen from the same side. Thus, owing to the anti-collinear momenta, the effects of interference cannot be seen on the screen.

To overcome this problem, one could work with beams that are neither collinear nor anti-collinear. For example, one could do a variant of the two-slit experiment in which slit 1 transmits a wave with frequency ω_1 , while slit 2 transmits a wave with frequency ω_2 . So let us generalize the analysis above to incorporate such possibilities as well. In a conventional approach, one deals with a wave function of the form (see the Appendix)

$$\varphi(\mathbf{x}, t) = \frac{1}{\sqrt{2}} [e^{-i\omega_1 t} \varphi_1(\mathbf{x}) + e^{-i\omega_2 t} \varphi_2(\mathbf{x})], \quad (19)$$

where $\varphi_{1,2}(\mathbf{x})$ are normalized such that $\int d^3x |\varphi_{1,2}(\mathbf{x})|^2 = 1$. The associated conventional probability density (see the Appendix) is $\rho(\mathbf{x}, t) = |\varphi(\mathbf{x}, t)|^2$. This gives

$$\rho(\mathbf{x}, t) = C(\mathbf{x}) + I(\mathbf{x}, t), \quad (20)$$

where

$$C(\mathbf{x}) = \frac{1}{2} [|\varphi_1(\mathbf{x})|^2 + |\varphi_2(\mathbf{x})|^2] \quad (21)$$

is the “classical” probability density and

$$I(\mathbf{x}, t) = \frac{1}{2} [e^{-i(\omega_1 - \omega_2)t} \varphi_1(\mathbf{x}) \varphi_2^*(\mathbf{x}) + e^{i(\omega_1 - \omega_2)t} \varphi_1^*(\mathbf{x}) \varphi_2(\mathbf{x})] \quad (22)$$

is the interference term.

In the approach based on the Klein-Gordon scalar product (see the Appendix), instead of (19) one deals with a differently normalized wave function

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{2}} \left[\frac{e^{-i\omega_1 t}}{\sqrt{2\omega_1}} \varphi_1(\mathbf{x}) + \frac{e^{-i\omega_2 t}}{\sqrt{2\omega_2}} \varphi_2(\mathbf{x}) \right], \quad (23)$$

which generalizes (13). This leads to

$$j_0(\mathbf{x}, t) = C(\mathbf{x}) + \alpha I(\mathbf{x}, t), \quad (24)$$

where

$$\alpha \equiv \frac{\omega_1 + \omega_2}{2\sqrt{\omega_1 \omega_2}}. \quad (25)$$

In the limit $\omega_1 = \omega_2$ we have $\alpha = 1$, so from (20) and (24) one recovers (11)

$$\rho = j_0 = C(\mathbf{x}) + I(\mathbf{x}), \quad (26)$$

where $I(\mathbf{x}) = [\varphi_1(\mathbf{x})\varphi_2^*(\mathbf{x}) + \varphi_1^*(\mathbf{x})\varphi_2(\mathbf{x})]/2$. This describes a usual stationary interference pattern. When $\omega_1 \neq \omega_2$, then both (20) and (24) describe a nonstationary interference pattern that oscillates with the frequency $\omega_1 - \omega_2$. Since $\alpha \neq 1$, these two patterns are different, which, in principle, could be distinguished experimentally. However, if $|\omega_1 - \omega_2|$ is large so that the oscillation is too fast to see it experimentally, then everything that can be seen is the time-averaged distribution

$$\langle \rho \rangle = \langle j_0 \rangle = C(\mathbf{x}), \quad (27)$$

that washes out all effects of interference and all differences between the two approaches. (More precisely, the time average $\langle j_0 \rangle = C(\mathbf{x})$ is observable if j_0 is non-negative. If it is negative at some regions then one should actually calculate $\langle |j_0| \rangle$ which may differ from $C(\mathbf{x})$. However, the typical distances at which the deviations from $C(\mathbf{x})$ occur are of the order $|\omega_1 - \omega_2|^{-1}$, which are small when $|\omega_1 - \omega_2|$ is large.) To see the oscillations one must have small $|\omega_1 - \omega_2|$ (i.e. $\omega_1 \simeq \omega_2$), but then $\alpha \simeq 1$ so the differences between (20) and (24) are difficult to see again.

3 Generalization to the many-particle case

Now we generalize the single-particle wave function $\psi(x)$ to the n -particle wave function $\psi(x_1, \dots, x_n)$ [5]. It satisfies n Klein-Gordon equations, one for each x_a , $a = 1, \dots, n$. Similarly, there are n conserved Klein-Gordon currents

$$j_{a\mu} = i\psi^* \overleftrightarrow{\partial}_{a\mu} \psi, \quad (28)$$

$$\partial_{a\mu} j_a^\mu = 0. \quad (29)$$

Eq. (29) is valid for each a , but these equations can also be summed to give

$$\sum_a \partial_{a\mu} j_a^\mu = 0. \quad (30)$$

In the Bohmian interpretation one postulates [10, 5, 11]

$$\frac{dx_a^\mu(s)}{ds} = j_a^\mu, \quad (31)$$

which determines n trajectories $x_a^\mu(s)$. These n trajectories in the 4-dimensional spacetime can also be viewed as one trajectory in the $4n$ -dimensional configuration spacetime.

Even though the Bohmian equation of motion (31) for n particles is nonlocal, it is completely relativistic covariant [11]. No *a priori* preferred foliation of spacetime is required. The functions $x_a^\mu(s)$ can be determined by a specification of $4n$ “initial” conditions $x_a^\mu(0)$. However, the price payed for this large symmetry is a smaller predictive power. Various choices of these “initial” conditions correspond to various choices of synchronization among the n particles [11].

To increase the predictive power of the theory, in the following we consider a different version of the theory, a version with a smaller symmetry. The Lorentz symmetry brakes by introducing a preferred foliation of spacetime specified by the timelike future-oriented unit normal vector $N^\mu(x)$. It satisfies $\nabla_\mu N^\mu = 0$, where ∇_μ is the covariant derivative generalizing the ordinary derivative ∂_μ to curved coordinates. Following [7], we introduce the n -vector

$$j_{\mu_1 \dots \mu_n}(x_1, \dots, x_n) = i^n \psi^* \overleftrightarrow{\partial}_{\mu_1} \cdots \overleftrightarrow{\partial}_{\mu_n} \psi, \quad (32)$$

where $\partial_{\mu_a} \equiv \partial/\partial x_a^{\mu_a}$. Now, analogously to the fermionic case studied in [12], we introduce n currents $j_{\mu_a}(x_1, \dots, x_n)$, $a = 1, \dots, n$, by contracting (32) $(n-1)$ times with the vector N^μ . For example, for $a = 1$,

$$j_{\mu_1}(x_1, \dots, x_n) = j_{\mu_1 \dots \mu_n}(x_1, \dots, x_n) N^{\mu_2}(x_2) \cdots N^{\mu_n}(x_n), \quad (33)$$

which satisfies $\nabla_{\mu_1} j^{\mu_1} = 0$. In (33) it is understood that all points lie at the same hypersurface of the preferred foliation. Thus, instead of (31), now the Bohmian particle trajectories are postulated to be

$$\frac{dx_a^\mu(s)}{ds} = j_{\mu a}. \quad (34)$$

From the results of [12] and [7] one finds that the probability density of particle positions on preferred hypersurfaces is

$$\tilde{p}(x_1, \dots, x_n) = |\tilde{N}^{\mu_1}(x_1) \cdots \tilde{N}^{\mu_n}(x_n) j_{\mu_1 \dots \mu_n}(x_1, \dots, x_n)|, \quad (35)$$

where

$$\tilde{N}^{\mu_a}(x_a) = |g_a^{(3)}(x_a)|^{1/2} N^{\mu_a}(x_a). \quad (36)$$

4 Generalization to strings

The Bohmian interpretation of string theory has been studied in [13, 14, 15, 16]. In the Bohmian context, strings have several advantages over particles or fields. First, bosons and fermions are treated symmetrically [15]. Second, the symmetry between bosons and fermions provides new insights on the origin of preferred foliation of spacetime at the level of effective field theory [15]. Third, the Bohmian equation of motion for strings automatically includes a continuous description of particle creation and destruction [15, 16]. (By contrast, to make Bohmian mechanics of particles compatible with particle creation and destruction, one is forced to make some artificial modifications of the theory [17, 18], [4, 19].)

Essentially, strings are obtained from many-particle systems through a replacement of the discrete label a by a continuous variable σ . Thus, instead of n coordinates x_a^μ we deal with a continuous set of string coordinates $x^\mu(\sigma)$. In [13, 14, 15, 16] we have studied the spacetime covariant version, specified by the wave functional $\psi[x(\sigma)] \equiv \psi[x]$. Here, by analogy with Sec. 3, we introduce a preferred foliation of spacetime specified by $N^\mu(x)$. (Of course, in string theory the number of space dimensions is not 3, but 25 in bosonic string theory and 9 in superstring theory [20, 21].) Next, we introduce the local symmetric hermitian functional-derivative operator

$$\hat{P}(\sigma) = i \left[N^\mu(x(\sigma)) \frac{\overrightarrow{\delta}}{\delta x^\mu(\sigma)} - \frac{\overleftarrow{\delta}}{\delta x^\mu(\sigma)} N^\mu(x(\sigma)) \right]. \quad (37)$$

Now, for bosonic strings, the string current is given by a generalization of (33)

$$j_\mu[x; \sigma] = i \psi^*[x] \frac{\overleftrightarrow{\delta}}{\delta x^\mu(\sigma)} \left\{ \prod_{\sigma'}^{(\sigma)} \hat{P}(\sigma') \right\} \psi[x]. \quad (38)$$

Here the notation $[x; \sigma]$ denotes a functional with respect to x and a function with respect to σ , while the product $\prod_{\sigma'}^{(\sigma)}$ denotes the product over all values of σ' except $\sigma' = \sigma$. In the superstring case it generalizes to

$$j_\mu[x; \sigma] = i \int [dM] \psi^*[x, M] \frac{\overleftrightarrow{\delta}}{\delta x^\mu(\sigma)} \left\{ \prod_{\sigma'}^{(\sigma)} \hat{P}(\sigma') \right\} \psi[x, M]. \quad (39)$$

where $M(\sigma)$ is an additional variable generalizing the spinor indices of particle wave functions [15]. Note that bosonic and fermionic string states are described by a single universal current

(39), which is a generalization of the Klein-Gordon (not the Dirac) current. Consequently, if superstring theory is correct, then, in the particle limit, the Bohmian particle trajectories of fermions are also described by a version of the Klein-Gordon current [15, 16]. The string current is conserved

$$\int d\sigma \frac{\delta j^\mu[x; \sigma]}{\delta x^\mu(\sigma)} = 0, \quad (40)$$

which implies that the local probability density that the string has the shape $x(\sigma)$ is given by a generalization of (35)

$$\begin{aligned} \tilde{p}[x] &= \left| \left\{ \prod_{\sigma'} |g^{(3)}(x(\sigma'))|^{1/2} \right\} N^\mu(x(\sigma)) j_\mu[x; \sigma] \right| \\ &= \left| \int [dM] \psi^*[x, M] \left\{ \prod_{\sigma'} \hat{\tilde{P}}(\sigma') \right\} \psi[x, M] \right|, \end{aligned} \quad (41)$$

where $\prod_{\sigma'}$ denotes the product over all values of σ' and $\hat{\tilde{P}}$ is obtained from \hat{P} by a replacement $N^\mu \rightarrow \tilde{N}^\mu$. This probability density is consistent with the Bohmian trajectories described by the functions $x^\mu(\sigma, s)$, satisfying the Bohmian equation of motion

$$\frac{dx^\mu(\sigma, s)}{ds} = j^\mu[x; \sigma], \quad (42)$$

which generalizes (34).

5 Conclusions

Even though the time component $j_0(\mathbf{x}, t)$ of the conserved Klein-Gordon current is not positive definite, the absolute value $|j_0(\mathbf{x}, t)|$ is. Therefore, as we have shown, this absolute value is the natural probability density of particle positions at time t . Further, we have shown that the fully relativistic covariant generalization of this is given by the probability density (7). Indeed, such probability density is locally conserved. The issue of global probability conservation is less trivial, but we have seen that the knowledge of the whole congruence of all Bohmian trajectories settles this issue as well. These results show that the Bohmian interpretation based on the Klein-Gordon current is fully relativistic covariant and fully predictive.

Further, although in some cases the predictions of this version of Bohmian mechanics may differ from the predictions of more conventional approaches to quantum theory, we have demonstrated that in practice such differences are difficult to observe. It seems that no already done experiment can be used to rule out this version of the Bohmian interpretation. Nevertheless, our results on practical measurability are not yet conclusive, so we challenge the readers to find an achievable experimental test that could confirm or falsify the predictions of this theory.

Finally, we have generalized our results to many-particle systems and strings. In agreement with [10, 12], we have found that a fully predictive theory with well-defined probabilities of particle positions cannot be constructed in a fully relativistic manner. Nevertheless, by introducing a preferred foliation of spacetime specified by the vector field of unit normals to hypersurfaces of the foliation, all equations can be written in a relativistic-covariant form.

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A Relativistic wave functions and their normalizations

A free scalar hermitian field operator can be expanded as [22]

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} [\hat{a}(\mathbf{k})e^{-ik \cdot x} + \hat{a}^\dagger(\mathbf{k})e^{ik \cdot x}], \quad (43)$$

where $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$ and $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. The destruction and creation operators satisfy

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}'). \quad (44)$$

The quantities $d^3k/2\omega_{\mathbf{k}}$ and $2\omega_{\mathbf{k}}\delta^3(\mathbf{k} - \mathbf{k}')$ are Lorentz invariant [22], which shows that the normalizations in (43) and (44) are manifestly Lorentz invariant.

Let $c(\mathbf{q})$ be an arbitrary c-number valued function normalized so that

$$\int \frac{d^3q}{(2\pi)^3} |c(\mathbf{q})|^2 = 1. \quad (45)$$

Such a function can be used to define the most general 1-particle state

$$|1\rangle = \int \frac{d^3q}{(2\pi)^3} c(\mathbf{q}) \frac{\hat{a}^\dagger(\mathbf{q})}{\sqrt{2\omega_{\mathbf{q}}}} |0\rangle, \quad (46)$$

where $|0\rangle$ is the vacuum, $\hat{a}(\mathbf{q})|0\rangle = 0$. Using (44) and (45), one finds that the normalization in (46) provides that $\langle 1|1\rangle = 1$. Following [22], the wave function associated with (46) can be defined as $\psi(x) = \langle 0|\hat{\phi}(x)|1\rangle$. Using (43), (44) and (46), this gives

$$\psi(x) = \int \frac{d^3q}{(2\pi)^3} c(\mathbf{q}) \frac{e^{-iq \cdot x}}{\sqrt{2\omega_{\mathbf{q}}}}. \quad (47)$$

The norm of this wave function is defined through the Klein-Gordon scalar product

$$(\psi, \psi') = i \int_{\Sigma} dS^\mu \psi^* \overleftrightarrow{\partial}_\mu \psi'. \quad (48)$$

When $\psi(x)$ and $\psi'(x)$ satisfy the Klein-Gordon equation, then (48) does not depend on the choice of the spacelike Cauchy hypersurface Σ . Therefore we choose Σ to be a hypersurface of constant Lorentz time-coordinate x^0 . This implies that the norm of (47) is

$$(\psi, \psi) = i \int d^3x \psi^* \overleftrightarrow{\partial}_0 \psi = 1, \quad (49)$$

where the identity

$$\int \frac{d^3x}{(2\pi)^3} e^{-i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{x}} = \delta^3(\mathbf{k} - \mathbf{k}') \quad (50)$$

and normalization (45) have been used.

The wave function (47) is not the only meaningful wave function that can be associated with the state (46). Another possibility is to introduce the wave function

$$\varphi(x) = \int \frac{d^3q}{(2\pi)^3} c(\mathbf{q}) e^{-iq \cdot x}. \quad (51)$$

Using (50) one finds that (51) has the property

$$\int d^3x \varphi^*(x) \varphi(x) = \int \frac{d^3k}{(2\pi)^3} \int d^3k' c^*(\mathbf{k}) c(\mathbf{k}') e^{i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})t} \delta^3(\mathbf{k} - \mathbf{k}'). \quad (52)$$

At first sight, this quantity seems time-dependent. However, owing to the δ -function we have $\mathbf{k} = \mathbf{k}'$. This implies that $e^{i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})t} = 1$, which removes the time-dependence. Consequently, (52) reduces to

$$\int d^3x \varphi^*(x) \varphi(x) = 1, \quad (53)$$

where (45) also has been used. This shows that the integral in (53) does *not* depend on time, i.e. that the quantity

$$\rho(\mathbf{x}, t) = \varphi^*(\mathbf{x}, t) \varphi(\mathbf{x}, t) \quad (54)$$

can be interpreted as the probability density of particle positions at time t [23]. On the other hand, the wave function $\varphi(\mathbf{x}, t)$ satisfies the Klein-Gordon equation and it is well-known that the integral on the left-hand side of (53) may depend on time when φ satisfies the Klein-Gordon equation (which is why a more complicated scalar product (48) has been introduced in the first place). So how is that possible that (53) does not depend on time? Where is the catch? The catch is [23] that (51) is *not* the most general solution of the Klein-Gordon equation. The most general solution involves a superposition of plane waves with both positive and negative frequencies, while (51) contains *only positive frequencies*. Indeed, if both positive and negative frequencies were involved, then (52) would also involve factors of the form $e^{i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'})t}$ and $e^{-i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'})t}$, which would *not* become time-independent when $\mathbf{k} = \pm \mathbf{k}'$. Thus, the restriction to the space of positive-frequency solutions allows us to use the conventional norm (53) and to interpret (54) as the conserved probability density.

Nevertheless, the Klein-Gordon norm (49) still has an advantage over the conventional norm (53). While (49) is Lorentz invariant, (53) is *not* Lorentz invariant. This is the main disadvantage of the probability density (54). Still, in a conventional operational interpretation of QM without hidden variables, this is not necessarily a problem if one simply postulates that the “preferred” Lorentz frame is the frame in which the observer is at rest. On the other hand, in hidden-variable interpretations in which physical quantities are assumed to make sense even without observers, such a subjective identification of the “preferred” Lorentz frame is unacceptable.

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